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ALMOST PERIODIC MEASURES ON THE TORUS

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To the memory of Shlomo Horowitz

ABSTRACT

Given a skew product flow $(T, \mathbf{T}^2)$ on the two torus, we construct a family of flows on $\mathbf{T}^3$ parametrized by elements of the circle $\mathbf{T}$. We show that under a certain condition on $(T, \mathbf{T}^2)$ almost every flow in this family is strictly ergodic. This is used to characterize minimal subsets of the flow $(T, \mathcal{P}(\mathbf{T}^2))$ induced by T on the space of probability measures on $\mathbf{T}^2$. Using a result of M. Herman, we give an example to show that this characterization does not hold for every T.

§1. Introduction and statement of the results

(a) *Strict ergodicity*

Let $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ be the one dimensional torus, and m the Lebesgue measure on $\mathbf{T}$. It will be sometimes convenient to identify an element of $\mathbf{R}$ with its image in $\mathbf{T}$.

Given a continuous function $h : \mathbf{T} \to \mathbf{T}$, and an irrational α, we consider the flow on the two dimensional torus $\mathbf{T}^2$ defined by the action of the homeomorphism

$$T = T_{\alpha,h} : (x, y) \to (x + \alpha, y + h(x)).$$

For every $n \in \mathbf{Z}$, the function $f_n, f_n(x, y) = e^{2\pi i n x}$, and the constant multiples of f_n, are eigenfunctions for T, with $e^{2\pi i n \alpha}$ as eigenvalue. Our assumption throughout this paper is that T has no other eigenfunction in the space $L^2(m^2)$.

It is easy to see that this assumption is equivalent to the following:

$$(*)\quad \begin{cases} \text{For every } \lambda \in \mathbf{C} \text{ for every } k \in \mathbf{Z}\backslash\{0\}, \text{ the functional equation} \\ f(x+\alpha)e^{2\pi i k h(x)} = \lambda f(x) \\ \text{has no non-zero measurable solution } f. \end{cases}$$

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It is known ([2]) that (*) implies that the Lebesgue measure m^2 is the unique T-invariant probability measure on $\mathbf{T}^2$, and since this measure assigns positive mass to non-empty open sets, T is strictly ergodic and minimal (i.e., having a unique invariant probability measure on the space, and such that every point has a dense orbit).

We shall show that whenever h is essential and satisfies a Lipschitz condition, then (*) is satisfied for every irrational α (lemma 2.4; see also [2, lemma 2.2]).

We define now two other flows. For each $\beta \in \mathbf{T}$, h and α being given as above, let $R_\beta : \mathbf{T}^3 \to \mathbf{T}^3$ and $S_\beta : \mathbf{T}^2 \to \mathbf{T}^2$ be defined by

$$R_\beta(x, y, z) = (x + \alpha, y + h(x), z + h(x + \beta)),$$

$$S_\beta(x, y) = (x + \alpha, y + h(x + \beta) - h(x)).$$

There exists a flow homomorphism from $(R_\beta, \mathbf{T}^3)$ to $(S_\beta, \mathbf{T}^2)$ given by the map $F : \mathbf{T}^3 \to \mathbf{T}^2$, $F(x, y, z) = (x, z - y)$.

The first result is the following.

THEOREM A. *If α and h satisfy (*), the flow $(R_\beta, \mathbf{T}^3)$, and hence also $(S_\beta, \mathbf{T}^2)$, are strictly ergodic for m-almost all $\beta \in \mathbf{T}$.*

Given α and h, let $\Gamma = \Gamma(\alpha, h)$ be the subset of those $\beta \in \mathbf{T}$ for which $(R_\beta, \mathbf{T}^3)$ is strictly ergodic, and let $\Delta = \mathbf{T} \backslash \Gamma$. From Theorem A, we have $m(\Delta) = 0$, when (*) is satisfied.

For $h(x) = x$, we have easily $\Delta = \mathbf{Q}\alpha + \mathbf{Q}$ ($\mathbf{Q}$ = rational numbers). This can be partially extended:

THEOREM B. *Let $h : \mathbf{T} \to \mathbf{T}$ be an essential map. If h is $C^{1+\varepsilon}$, $\varepsilon > 0$, we have for almost all α: $\Delta(\alpha, h) = \mathbf{Q}\alpha + \mathbf{Q}$.*

It is known that strict ergodicity implies well distribution properties for sequences generated by the flow.

If we define

$$h_n(x) = \begin{cases} h(x) + h(x + \alpha) + \cdots + h(x + (n-1)\alpha), & n > 0 \\ 0, & n = 0 \\ -h(n - \alpha) - h(x - 2\alpha) - \cdots - h(x - n\alpha), & n < 0 \end{cases}$$

and $H_n(x) = h_n(x) - h_n(0)$, $n \in \mathbf{Z}$, we have:

$$T^n(x, y) = (x + n\alpha, y + h_n(x)), \quad \text{and} \quad S^n_\beta(0,0) = (n\alpha, H_n(\beta)).$$

Thus, Theorem A implies as a

COROLLARY. *If* (*) *is satisfied, for* m*-almost all* β, *the sequence* $\{H_n(\beta), n \in \mathbf{Z}\}$ *is well distributed.*

From Theorem B, it follows:

If h *is a* $C^{1+\varepsilon}$ *essential map, for* m*-almost all* α, *the sequence* $\{H_n(\beta), n \in \mathbf{Z}\}$ *is well distributed, when* $\beta \notin \mathbf{Q}\alpha + \mathbf{Q}$.

(b) *Almost periodicity of measures*

We are also interested here in some simple topological properties of the flow $(T, \mathcal{P}(\mathbf{T}^2))$ induced by T on the space $\mathcal{P}(\mathbf{T}^2)$ of probability measures on $\mathbf{T}^2$ equipped with the weak * topology. (We still denote by T the homeomorphism induced on $\mathcal{P}(\mathbf{T}^2)$ by $T = T_{\alpha,h}$.)

A measure $\mu \in \mathcal{P}(\mathbf{T}^2)$ is called *almost periodic* (a.p.) (for $(T, \mathcal{P}(\mathbf{T}^2))$, if its T-orbit has a closure in $\mathcal{P}(\mathbf{T}^2)$, denoted by $\overline{O(\mu)}$, which is a minimal set (i.e. the orbit of every $\mu_1 \in \overline{O(\mu)}$ is dense in $\overline{O(\mu)}$).

For any measure $\nu \in \mathcal{P}(\mathbf{T})$, we have easily that the orbit closure of ν under the action of the rotation $x \to x + \alpha$ on $\mathcal{P}(\mathbf{T})$ is a minimal set. This implies that, for every $\nu \in \mathcal{P}(\mathbf{T})$, the product measure $\nu \times m$ is an a.p. point of the flow $(T, \mathcal{P}(\mathbf{T}^2))$.

Let $\pi : \mathbf{T}^2 \to \mathbf{T}$ be the projection $\pi(x, y) = x$, and also $\pi : \mathcal{P}(\mathbf{T}^2) \to \mathcal{P}(\mathbf{T})$ the induced map on the space of probability measures. For $\mu \in \mathcal{P}(\mathbf{T}^2)$ with $\pi(\mu) = \nu$, we write $\nu = \nu_c + \nu_d$, where $\nu_d = \Sigma\, a_i \delta_{x_i}$ is the purely discontinuous part of ν and ν_c is its continuous part (i.e. ν_c has no atoms). Let μ_i be the restriction of μ to $\pi^{-1}(x_i)$ and write $\mu = \mu' + \mu''$ where $\mu'' = \Sigma\, \mu_i$. For an arbitrary irrational α and the function $h(x) = x$, the following characterization of a.p. measure of $(T, \mathcal{P}(\mathbf{T}^2))$ was given in [3].

A measure $\mu \in \mathcal{P}(\mathbf{T}^2)$ is a.p. iff $\mu' = \nu_c \times m$. Each orbit closure in $\mathcal{P}(\mathbf{T}^2)$ contains a unique minimal set. We prove here the following theorems.

THEOREM C. *Let* α *and* h *satisfy* (*). *Let* $\mu \in \mathcal{P}(\mathbf{T}^2)$ *with* $\nu = \pi(\mu)$ *an absolutely continuous measure (with respect to* m*). Then* $\nu \times m \in \bar{O}(\mu)$. *If in addition* μ *is a.p. then* $\mu = \nu \times m$.

THEOREM D. *Suppose that $\Delta(\alpha, h)$ is countable. Then a measure $\mu \in \mathcal{P}(\mathbf{T}^2)$ is a.p. iff $\mu' = \nu_c \times m$. Each orbit closure in $\mathcal{P}(\mathbf{T}^2)$ contains a unique minimal set.*

Is $\Delta(\alpha, h)$ always countable? The answer is no; we produce an irrational α and a continuous function h for which the assumption (*) holds and yet $\Delta(\alpha, h)$ (which is of measure zero) is uncountable. We also show that for the corresponding flow $(T, \mathbf{T}^2)$, there exists an a.p. $\mu \in \mathcal{P}(\mathbf{T}^2)$ for which $\pi(\mu) = \nu$ is continuous and yet $\mu \neq \nu \times m$.

Analogous results about the topological behaviour of $\{H_n(x)\}$ and the character of almost periodic closed subsets of $(T, \mathbf{T}^2)$ were obtained in [3].

§2. The set $\Delta(\alpha, h)$. Proofs of Theorems A and B

2.1. The following proposition will be used in Theorems A, C, D. It is classical and we omit the proof.

Consider the product flow $(T \times T, \mathbf{T}^2 \times \mathbf{T}^2)$ given by

$$(T \times T)((x, y), (z, w)) = ((x + \alpha, y + h(x)), (z + \alpha, w + h(z))).$$

Let $\mathcal{I}$ denote the subspace of $L_2(m^4)$ of $T \times T$-invariant functions.

PROPOSITION. *If α and h satisfy (*), $\mathcal{I}$ is spanned by the functions of the form $(x, y, z, w) \to e^{2\pi i k(x-z)}$, $k \in \mathbf{Z}$.*

2.2. PROOF OF THEOREM A. Since $(R_\beta, \mathbf{T}^3)$ is a group extension of the strictly ergodic flow $(T, \mathbf{T}^2)$, it is enough to show that $(R_\beta, \mathbf{T}^3, m^3)$ is ergodic [2, lemma 2.1].

Let $(l, p, k) \in \mathbf{Z}^3$. Let f be defined on $\mathbf{T}^4$ by

$$f(x, y, z, w) = e^{2\pi i (lx + py + kw)}.$$

By the ergodic theorem, we have

$$\frac{1}{N} \sum_{n=0}^{N-1} (T \times T)^n f \to E(f \mid \mathcal{I}) \qquad m^4\text{-a.e.}$$

Here $E(\cdot \mid \mathcal{I})$ is the projection of $L_2(m^4)$ onto $\mathcal{I}$. Now if $(l, p, k) \neq (0, 0, 0)$ then by Proposition 2.1 f is orthogonal to $\mathcal{I}$ and $E(f \mid \mathcal{I}) = 0$. Thus

$$\lim \frac{1}{N} \sum_{n=0}^{N-1} \exp\{2\pi i [l(x + n\alpha) + p(y + h_n(x) + k(w + h_n(z))]\} = 0$$

for m^4 almost all x, y, w and z. Writing $\beta = z - x$ we conclude that for almost all β and $g(x, y, w) = g(x, y, x + \beta, w)$

$$\lim \frac{1}{N} \sum_{n=0}^{N-1} R_\beta^n g(x, y, w) = 0 = \int g dm^3 \qquad m^3\text{-a.e.}$$

Since the functions $g(x, y, w)$ which correspond to $(l, p, k) \neq 0$ together with the constant functions span $L_2(m^3)$, we conclude that $(R_\beta, \mathbf{T}^3)$ is ergodic. The strict ergodicity of $(S_\beta, \mathbf{T}^2)$ follows since the latter is a factor of the former flow. □

2.3. Remarks. (a) Suppose that α and h do not satisfy (*). Then, for some $k \in \mathbf{Z} - \{0\}$ and $\lambda \in \mathbf{C}$, $|\lambda| = 1$, there exists a non-zero measurable function f such that

$$f(x + \alpha) e^{2\pi i k h(x)} = \lambda f(x).$$

Define $g_\beta(x) = f(x)/f(x + \beta)$. We have

$$g_\beta(x) = e^{2\pi i k [h(x) - h(x+\beta)]} g_\beta(x + \alpha),$$

for every $\beta \in \mathbf{T}$. Therefore $(S_\beta, \mathbf{T}^2)$, and hence also $(R_\beta, \mathbf{T}^3)$, are not strictly ergodic (cf. [2, lemma 2.1]), and we have $\Delta(\alpha, h) = \mathbf{T}$.

This shows that we have (*) iff $m(\Delta) = 0$, and that Δ is either all of $\mathbf{T}$ or a set of measure zero.

These results can be proved directly, using the fact that Δ is a measurable subgroup of $\mathbf{T}$.

(b) The results in [3] were obtained under the assumption that the flow $(T, \mathbf{T}^2)$ is not equicontinuous rather than the *a priori* stronger condition that the only continuous eigenfunctions of $(T, \mathbf{T}^2)$ are the functions $e^{2\pi i k x}$ $(k \in \mathbf{Z})$, which is the topological analogue for our condition (*). However, lemma 2.2 of [3] shows that the weaker condition implies the stronger one.

2.4. Given a continuous function $h : \mathbf{T} \to \mathbf{T}$, there exists a continuous "lift" $\tilde{h} : [0, 1] \to \mathbf{R}$ (i.e. $\tilde{h}(t) = h(t) \pmod 1$). The integer $d = \tilde{h}(1) - \tilde{h}(0)$ depends only on h and is called the *index* of h. Clearly h is essential (i.e. non-homotopic to a constant) iff $d \neq 0$. Let $\tilde{g} : [0, 1] \to \mathbf{R}$ be defined by $\tilde{g}(x) = \tilde{h}(x) - dx$. Then $\tilde{g}(1) - \tilde{g}(0) = 0$ and $\tilde{g} : \mathbf{T} \to \mathbf{R}$ is continuous. Our next goal is to prove Theorem B. The proof of the following lemma is essentially that of lemma 2.2 of [2].

Lemma. *Let $h : \mathbf{T} \to \mathbf{T}$ be an essential map of index $d \neq 0$, and suppose that for* *all* $x, x' \in \mathbf{T}$, $|h(x) - h(x')| < M|x - x'|$. *Then condition* $(*)$ *is satisfied for every irrational* $\alpha \in \mathbf{T}$.

2.5. The next lemma is due to M. Herman.

LEMMA. *Let* $\varphi : \mathbf{T} \to \mathbf{R}$ *be of class* $C^{1+\varepsilon}$. *Let* $\beta = \int_{\mathbf{T}} \varphi(x)dx$. *Then for almost all* $\alpha \in \mathbf{T}$ *the functional equation* $f(x + \alpha) = e^{2\pi i \varphi(x)} f(x)$ *has a non-zero measurable solution* f *iff* $\beta \in \mathbf{Z}\alpha + \mathbf{Z}$.

PROOF. Consider $\psi(x) = \varphi(x) - \beta$. Using Fourier series, a formal solution u of the (additive) functional equation

$$u(x + \alpha) = \psi(x) + u(x)$$

can be defined. For every ε', $0 < \varepsilon' < \varepsilon$, for almost all α, the solution u can be shown to be ε'-differentiable.

Taking the exponential, we get

$$e^{2\pi i u(x+\alpha)} = e^{-2\pi i \beta} e^{2\pi i \varphi(x)} e^{2\pi i u(x)}.$$

Therefore, the given functional equation has a solution iff $e^{-2\pi i \beta}$ is an eigenvalue of the rotation by α, i.e. $\beta \in \mathbf{Z}\alpha + \mathbf{Z}$.

2.6. PROOF OF THEOREM B. Using Fourier coefficients, it can be shown that $\beta \in \Delta(h, \alpha)$ iff, for some $(k, l) \in \mathbf{Z}^2 - \{(0,0)\}$, there exists a non-zero measurable solution f of the equation

$$f(x + \alpha) e^{2\pi i [kh(x+\beta) + lh(x)]} = f(x).$$

For $k \neq -l$, the function $x \to kh(x + \beta) + lh(x)$ is an essential function, which satisfies a Lipschitz condition. By Lemma 2.4, there exists no non-zero measurable solution of the above equation. Thus $k = -l$, and we have, with the notation used in 2.4,

$$f(x + \alpha) e^{2\pi i \varphi(x)} = f(x),$$

where $\varphi(x) = k[\tilde{g}(x + \beta) - \tilde{g}(x) + d\beta]$. Applying Lemma 2.5 to φ, for a.a. α, we get $kd\beta \in \mathbf{Z}\alpha + \mathbf{Z}$, i.e. $\beta \in \mathbf{Q}\alpha + \mathbf{Q}$.

§3. Almost periodic measures

3.1. PROPOSITION. *Suppose* α *and* h *satisfy* $(*)$, *and let* $\mu \in \mathcal{P}(\mathbf{T}^2)$ *be* *absolutely continuous* (*with respect to* m^2). *Let* $\pi(\mu) = \nu$; *then* $\nu \times m \in \bar{O}(\mu)$. *If in addition* μ *is a.p. then* $\mu = \nu \times m$.

PROOF. We show that for every $(k,l) \in \mathbf{Z}^2$ with $l \neq 0$

$$\lim \frac{1}{N} \sum_{n=0}^{N-1} |\widehat{T^n\mu}(k,l)|^2 = 0, \tag{1}$$

where $\hat{}$ denotes Fourier transform.

This implies that there exists a sequence n_j for which

$$\lim \widehat{T^{n_j}\mu}(k,l) = 0, \qquad \forall (k,l) \in \mathbf{Z}^2, \quad l \neq 0.$$

We can assume that $\lim T^{n_j}\mu = \eta$ exists and let $\lim T^{n_j}\nu = \theta$. Then $\hat{\eta}(k,l) = 0$ when $l \neq 0$ and $\hat{\eta}(k,0) = \hat{\theta}(k)$. Thus $\eta = \theta \times m$ and since $\bar{O}(\nu)$ is minimal, we have $\nu \times m \in \bar{O}(\theta \times m) \subset \bar{O}(\mu)$. Thus it suffices to show that (1) holds. Let $(k,l) \in \mathbf{T}^2$ with $l \neq 0$ be given and let $g(x,y) = e^{2\pi i(kx+ly)}$. Suppose $d\mu = f(x,y)dm^2$ where $f \in L_1(m^2)$. Then

$$\begin{aligned}\widehat{T^n\mu}(k,l) &= \int\int g(x+n\alpha, y+h_n(x))f(x,y)dxdy \\ &= \langle T^n g, \bar{f}\rangle\end{aligned}$$

and

$$|\widehat{T^n\mu}(k,l)|^2 = \langle (T \times T)^n (g \otimes \bar{g}), \bar{f} \otimes f\rangle.$$

By the ergodic theorem

$$\lim \frac{1}{N} \sum_{n=0}^{N-1} (T \times T)^n (g \otimes \bar{g}) = E(g \otimes \bar{g} \mid \mathcal{I}) \qquad m^4\text{-a.e.}$$

Since $l \neq 0$, $g \otimes \bar{g}$ is orthogonal to $\mathcal{I}$ and $E(g \otimes \bar{g} \mid \mathcal{I}) = 0$. This proves (1). The last assertion is clear. □

THEOREM C. *Let* α *and* h *satisfy* (*). *Let* $\mu \in \mathcal{P}(\mathbf{T}^2)$ *and assume that* $\nu = \pi(\mu)$ *is absolutely continuous* (*with respect to* m). *Then* $\nu \times m \in \bar{O}(\mu)$. *If in addition* μ *is* a.p. *then* $\mu = \nu \times m$.

PROOF. Let u be a probability measure on $\mathbf{T}$ and let $\theta \in \mathcal{P}(\mathbf{T}^2)$. Define $u * \theta \in \mathcal{P}(\mathbf{T}^2)$ by

$$\int_{\mathbf{T}^2} f(x,y)d(u * \theta) = \int_{\mathbf{T}^2}\int_{\mathbf{T}} f(x, y+z)du(z)d\theta(x,y).$$

It is easy to check that $\theta \to u * \theta$ is a homomorphism of $(T, \mathcal{P}(\mathbf{T}^2))$ into itself. Since $\bar{O}(\mu)$ contains a minimal subset we can assume that μ itself is a.p. Let $\{u_n\}$ be a sequence of absolutely continuous measures on $\mathbf{T}$ which converges to δ_0, the point mass at zero. Then for each n, $u_n * \mu$ is an a.p. and absolutely continuous measure on $\mathbf{T}^2$ with $\pi(u_n * \mu) = \nu$. Hence, by Proposition 3.1, $u_n * \mu = \nu \times m$. But $\lim u_n * \mu = \delta_0 * \mu = \mu$ and we conclude that $\mu = \nu \times m$. □

We recall the following notation which was introduced in section 1. For a measure $\mu \in \mathcal{P}(\mathbf{T}^2)$ with $\pi(\mu) = \nu$ we let $\nu = \nu_c + \nu_d$ be the decomposition of ν into continuous and purely discontinuous parts. Suppose $\nu_d = \Sigma\, a_i \delta_{x_i}$ ($x_i \in \mathbf{T}$, $a_i > 0$) and let μ_i be the restriction of μ to $\pi^{-1}(x_i)$. Write $\mu'' = \Sigma\, \mu_i$ and $\mu = \mu'' + \mu'$. One can easily show that $\mu''/\mu''(\mathbf{T}^2)$ is an a.p. element of $(T, \mathcal{P}(\mathbf{T}^2))$, [3].

THEOREM D. *Suppose that $\Delta(\alpha, h)$ is countable. Then a measure $\mu \in \mathcal{P}(\mathbf{T}^2)$ is a.p. iff $\mu' = \nu_c \times m$. Each orbit closure in $(T, \mathcal{P}(\mathbf{T}^2))$ contains a unique minimal set.*

PROOF. Since $\mu''/\mu''(\mathbf{T}^2)$ and $\nu_c \times m/(\nu_c \times m)(\mathbf{T}^2)$ are a.p. the condition is sufficient. Moreover when proving necessity we can assume that $\mu = \mu'$. Thus our assumption is that $\pi(\mu) = \nu$ is continuous. As in the proof of Proposition 3.1 it suffices to show that for every k and $l \neq 0$,

$$\lim \frac{1}{N} \sum_{n=0}^{N-1} |\widehat{T^n \mu}(k, l)|^2 = 0.$$

Put $g(x, y) = e^{2\pi i(kx + ly)}$, then

$$|\widehat{T^n \mu}(k, l)|^2 = \langle (T \times T)^n g \otimes g, \mu \times \bar{\mu} \rangle.$$

Let $f(x, y, z) = g \otimes \bar{g}(x, y, x + \beta, w) = e^{2\pi i[l(y-w) - k\beta]}$. For $\beta \notin \Delta$, by strict ergodicity of $(R_\beta, \mathbf{T}^3)$, we have, since $l \neq 0$, for every $(x, y, z) \in \mathbf{T}^3$:

$$\lim \frac{1}{N} \sum_{n=0}^{N-1} R_\beta^n f(x, y, z) = \int_{\mathbf{T}^3} f dm^3 = 0.$$

Therefore, outside a set $B \subset \{(x, y, z, w) : z - x \in \Delta\}$,

$$\lim \frac{1}{N} \sum_{n=0}^{N-1} (T \times T)^n g \otimes \bar{g}(x, y, z, w) = 0.$$

Since Δ is assumed to be countable, and ν is continuous, we have $(\mu \times \mu)(B) = 0$. By Lebesgue's convergence theorem, we conclude

$$\lim \frac{1}{N} \sum_{n=0}^{N-1} \int (T \times T^n) g \otimes \bar{g} d\mu \times d\mu = 0.$$

§4. A counter example

In this section we produce an irrational α and a continuous function $h : \mathbf{T} \to \mathbf{T}$ such that (a) α and h satisfy (*), (b) $\Delta(\alpha, h)$ is uncountable, (c) Theorem D does not hold for $(T, \mathbf{T}^2)$; i.e. there exists an a.p. $\mu \in \mathcal{P}(\mathbf{T}^2)$ with $\nu = \pi(\mu)$ continuous and $\mu \neq \nu \times m$.

As in [2] define a sequence of integers v_k by $v_1 = 1$ and $v_{k+1} = 2^{v_k} + v_k + 1$. Set $n_k = 2^{v_k}$ and $\alpha = \sum_{k=1}^{\infty} n_k^{-1}$. Then

$$|n_k\alpha - [n_k\alpha]| < \frac{2 \cdot 2^{v_k}}{2^{v_{k+1}}} = 2^{-n_k}$$

where $[\cdot]$ denotes integral part. Let $n_{-k} = -n_k$ and write

$$h(x) = \sum_{k \neq 0} (e^{2\pi i n_k \alpha} - 1)e^{2\pi i n_k x}.$$

The sequence $\{n_k\}_{k=1}^{\infty}$ is lacunary and $h(x)$ is infinitely differentiable. For a real number t let $h' = t \cdot h$.

4.1. Proposition. *There exists a t, $0 \leqq t \leqq 1$, for which α and h' satisfy (*).*

Proof. By Remark 2.3(a) it suffices to show that there exist t and β such that for every $l \neq 0$ the equation

$$(1) \qquad g(x+\alpha)e^{2\pi i l[h'(x+\beta)-h'(x)]} = g(x)$$

does not admit a non-zero measurable solution g. By a result of J. P. Conze [1], if the additive equation

$$(2) \qquad l[h(x+\beta)-h(x)] = \psi(x+\alpha) - \psi(x)$$

admits no measurable solution ψ then for almost every t equation (1) admits no measurable solution. It is therefore enough to show that (2) admits no measurable solution for some β.

If such a measurable solution exists, and belongs to L^2, we have from (2) the following equation for the Fourier coefficients:

$$l(e^{2\pi i n_k \beta} - 1)\hat{h}(n_k) = (e^{2\pi i n_k \alpha} - 1)\hat{\psi}(n_k)$$

or

$$\hat{\psi}(n_k) = l(e^{2\pi i n_k \beta} - 1).$$

Now if $\beta \in \mathbf{T}$ is such that $\Sigma |e^{2\pi i n_k \beta} - 1|^2 = \infty$ then we can conclude that (2) admits no $L^2(m)$ solution. Since $\{n_k\}$ is lacunary it follows from a result of M. Herman, [4], that (2) admits no measurable solution as well. The proof is completed. □

4.2. PROPOSITION. *For every t, $\Delta(\alpha, h^t)$ is uncountable.*

PROOF. Let $\beta \in \mathbf{T}$ satisfy $\Sigma_{k\neq 0}|e^{2\pi i n_k \beta} - 1|^2 < \infty$ then (2) above admits a solution (for $l = 1$):

$$\psi(x) = \sum_{k\neq 0} (e^{2\pi i n_k \beta} - 1)e^{2\pi i n_k x}.$$

Hence for every t, (1) admits a solution and $\beta \in \Delta(\alpha, h^t)$. The condition $\Sigma_{k\neq 0}|e^{2\pi i n_k \beta} - 1|^2 < \infty$ is satisfied by an uncountable number of $\beta \in \mathbf{T}$ and hence $\Delta(\alpha, h^t)$ is uncountable. □

4.3. PROPOSITION. *Fix a t_0, $0 \leqq t_0 \leqq 1$ for which α, h^{t_0} satisfy (*) and let $(T, \mathbf{T}^2)$ be the corresponding flow. Then there exists an* a.p. *measure $\mu \in \mathcal{P}(\mathbf{T}^2)$ with $\nu = \pi(\mu)$ continuous and $\mu \neq \nu \times m$.*

PROOF. Let $\Omega = \{\omega \in \mathbf{T} : \omega = \Sigma\, \varepsilon_k n_k^{-1};\ \varepsilon_k = 0, 1\}$; then Ω is a closed subset of $\mathbf{T}$ which is homeomorphic to a Cantor set. Define a function $D : \Omega \to \mathbf{R}$ by

$$D(\omega) = \sum_{k\neq 0} |e^{2\pi i n_k \omega} - 1|.$$

Clearly D is continuous. We notice that

$$H_n(x) = \sum_{k\neq 0} (e^{2\pi i n_k n\alpha} - 1)(e^{2\pi i n_k x} - 1) \qquad (x \in \mathbf{T}),$$

and therefore for every $n \in \mathbf{Z}$

$$|H_n(\omega)| \leqq 2\sum |e^{2\pi i n_k \omega} - 1| = 2D(\omega).$$

Let ν be an arbitrary continuous measure on Ω with $\mathrm{Supp}(\nu) = \Omega$ and put $\eta = \nu \times \delta_0$. Let μ be an a.p. measure in $\bar{O}(\eta)$ such that $\pi(\mu) = \nu$. There exists a sequence of integers $\{n_j\}$ such that $\lim T^{n_j}\eta \to \mu$ and $n_j\alpha \to 0$. Clearly

$$\overline{\lim}\,[\mathrm{Supp}(T^{n_j}\eta)] \supset \mathrm{Supp}(\mu).$$

We observe that

$$\mathrm{Supp}(T^{n_j}\eta) = \{(\omega + \eta_j\alpha, h_{n_j}(\omega)) : \omega \in \Omega\}$$
$$= \{(\omega + n_j\alpha, H_{n_j}(\omega) + h_{n_j}(0)) : \omega \in \Omega\}.$$

Now let $(x, y) \in \overline{\lim}[\mathrm{Supp}(T^{n_j}\eta)]$, then for some sequence $\{\omega_j\} \subset \Omega$

$$(x, y) = \lim T^{n_j}(\omega_j, 0)$$
$$= \lim(\omega_j + n_j\alpha, H_{n_j}(\omega_j) + h_{n_j}(0)).$$

Without loss of generality we can assume that $y_0 = \lim h_{n_j}(0)$ exists and then $x = \lim \omega_j$ and

$$|y - y_0| = |\lim H_{n_j}(\omega_j)| \leqq \lim 2D(\omega_j) = 2D(x).$$

Thus we have

$$\mathrm{Supp}(\mu) \subset \{(\omega, y) : |y - y_0| \leqq 2D(\omega)\}.$$

In particular $\mathrm{Supp}(\mu) \neq \Omega \times \mathbf{T}$ and $\mu \neq \nu \times m$. □

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